Homework 3

Date: Sep 14, 2021 — Algebra & Computation-F21
Instructor: Mrinal Kumar — Due: Oct 5, 2021, 5 pm

Instructions

- It is slightly preferred that you type your homeworks up in LaTeX. In case you turn in scans of handwritten notes, please make sure that they are legible.
- Discussion on the problems with other members of the class is permitted and to an extent, even encouraged. But, you *must* write the solutions on your own. You *must* also acknowledge any discussions you might have had with others separately for every problem.
- Please do not look up solutions on the internet or in other references. In case you use any sources outside the notes for this course, again properly acknowledge them.
- To get the most out of the problem sets, you are encouraged to think about the problems on your own before discussing them with others, consulting the references or looking at the hints (which some of the problems might have).

Problems

1. **(10 points)**[1] Let $S_{d,n} = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{N} \cup \{0\}, \sum_i a_i \leq d\} \subseteq \mathbb{C}^n$. Show that for every non-zero n variate polynomial $f(x_1 x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ of total degree at most d, there exists a $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in S_{d,n}$ such that $f(\mathbf{b}) \neq 0$.

2. **(10 points)** Let $p_1, p_2, \ldots, p_n \in \mathbb{N}$ be n distinct prime numbers. Let $T_{d,n} = \{(p_1^i, p_2^i, \ldots, p_n^i) : i \in \{0, 1, 2, \ldots, \binom{n+d}{d} - 1\}\}$. Show that for every non-zero n variate degree d polynomial $f(x_1 x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, there exists a point $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in T_{d,n}$ such that $f(\mathbf{b}) \neq 0$.

3. **(10 points)** Let $\mathbb{F}$ be any field. A polynomial is said to be multilinear if its degree in every variable is at most one. Show that for every non-zero multilinear polynomial $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$, there is an $\mathbf{a} \in \{0, 1\}^n$ such that $P(\mathbf{a}) \neq 0$. [2]

4. **(10 points)** For $i \in \{1, 2, \ldots, k\}$, let $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$ be disjoint n-tuples of variables. For n variate polynomials $f_1(\mathbf{y}), \ldots, f_k(\mathbf{y}) \in \mathbb{C}[y_1, y_2, \ldots, y_n]$, let M be the $k \times k$ matrix such that $M_{i,j} = f_i(\mathbf{x}_j)$.

   Show that $f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_k(\mathbf{y})$ are linearly independent over $\mathbb{C}$ if and only if the determinant of M is non-zero.

[1] Throughout this homework, $\mathbb{C}$ is the field of complex numbers.
[2] Notice the difference between the scenario above and the polynomial identity lemma that we saw in the class.