Problem Set-1

Date : Sep 22 CS229r :Arithmetic Circuits
Due : Oct 13 Fall 2017, Harvard University

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1. Let P be a homogeneous polynomial of degree d which can be computed by an algebraic branching program of size s. What is the size of the smallest *homogeneous* algebraic branching program that you can come up with, which computes P ?

2. A homogeneous polynomial $P \in \mathbb{F}[X_1, X_2, \ldots, X_n]$ of degree d is said to be *set-multilinear* if P is multilinear and there exists a partition of $\{X_1, X_2, \ldots, X_n\}$ into $S_1, S_2, \ldots, S_d$ such that every monomial present in P contains exactly one variable from each S_i. We say that a circuit C is set-multilinear if the polynomial computed at every gate in C is set-multilinear.

   If a set multilinear polynomial P of degree d is computable by an arbitrary circuit of size s, then show that P can be computed by a set-multilinear circuit of size at most $O(2^d \cdot s)$.

3. If a homogeneous polynomial P of degree d is computable by a formula Φ of size s, then show that P is computable by a homogeneous formula Φ' of size at most $O\left(\binom{d+5\log s+1}{d} \cdot s\right)$.

4. Let P be a homogeneous polynomial of degree d which is computable by a homogeneous formula Φ of size s. Then, show that P can be expressed as

$$P = \sum_{i=1}^{s} Q_{i1} \cdot Q_{i2} \cdots Q_{ir} \,,$$

   where

   - $r = \Theta(\log d)$.
   - Each $Q_{i,j}$ is a homogeneous polynomial, which can be computed by a formula of size at most s.
   - For every i, $\sum_{j=1}^{r} \deg(Q_{i,j}) = d$
   - For every i, j, $\left(\frac{1}{3}\right)^j \cdot d < \deg(Q_{i,j}) \leq \left(\frac{2}{3}\right)^j \cdot d$.

   A decomposition of this type is called a *log-product* decomposition of a homogeneous formula, and will be very useful to us when we study multilinear formula lower bounds.

5. Recall the polynomial $IMM(n, d)$ we discussed in the class : it is the $(1, 1)$ entry in the product of $n \times n$ matrices $M_1, M_2, \ldots, M_d$, where $M_i(j_1, j_2) = X_{i,j_1,j_2}$. Show that $IMM(n, d)$ can be computed by a homogeneous depth-4 circuit (also written as $\sum\prod\sum\prod$ circuit) of size at most $n^{O(\sqrt{d})}$. What can you say about circuits of depth Δ computing it ?