On the power of border of depth-3 arithmetic circuits

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Abstract

We show that over the field of complex numbers, *every* homogeneous polynomial of degree d can be approximated (in the border complexity sense) by a depth-3 arithmetic circuit of top fan-in at most 2. This is quite surprising since there exist homogeneous polynomials P on n variables of degree 2, such that any depth-3 arithmetic circuit computing P must have top fan-in at least $\Omega(n)$.

As an application, we get a new tradeoff between the top fan-in and formal degree in an approximate analog of the celebrated depth reduction result of Gupta, Kamath, Kayal and Saptharishi [GKKS16, Tav15]. Formally, we show that if a degree d homogeneous polynomial P can be computed by an arithmetic circuit of size $s \geq d$, then for every $t \leq d$, P is in the border of a depth-3 circuit of top fan-in $s^{O(t)}$ and formal degree $s^{O(d/t)}$. To the best of our knowledge, the upper bound on the top fan-in in the original proof of [GKKS16] is always at least $s^{\Omega(\sqrt{d})}$, regardless of the formal degree.

1 Introduction

An arithmetic circuit on variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ over a field $\mathbb{F}$ is a directed acyclic graph with leaves labeled by variables in $\mathbf{x}$ and constants in $\mathbb{F}$ and internal vertices labeled by sum $(+)$ or product $(\times)$. Such a circuit provides a natural succinct representation for multivariate polynomials in $\mathbb{F}[\mathbf{x}]$. In this paper, the principle object of interest will be arithmetic circuits of depth-3, which we now define.

A depth-3 circuit (denoted by $\sum\prod\sum$) is an arithmetic circuit whose internal gates are arranged in three layers of alternating sums and products with the top layer being a sum. Such a circuit gives a representation of a polynomial as a sum of products of affine forms. Some of the parameters of interest of a $\sum\prod\sum$ circuit C are the fan-in of the top layer, which we call the top fan-in of C, the maximum of the degrees of its product gates, which we call the formal degree of C, and the total number of wires in the circuit which is called the size of the circuit. Note that for a depth-3 circuit, the formal degree is always at least as large as the size.

A crucial point to note is that the formal degree of a circuit C can be much much larger than the degree of the polynomial computed by C. Such a $\sum\prod\sum$ circuit computes polynomials of really high degree by taking products of affine forms, and then takes a linear combination of many such high degree polynomials to *efficiently* compute a much lower degree polynomial via cancellations. One classic example of this is a result of Ben-Or [NW97] who showed that over large enough fields, for every degree $d \in [n]$, the elementary symmetric polynomial of degree d in variables $\mathbf{x}$ defined as

$$\mathrm{SYM}_d(\mathbf{x}) = \sum_{S \in \binom{[n]}{d}} \prod_{j \in S} x_j \, ,$$

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can be computed by a $\sum\prod\sum$ circuit with top fan-in $n+1$ and formal degree n. In sharp contrast, Nisan and Wigderson [NW97] had earlier shown that any $\sum\prod\sum$ circuit computing $\mathrm{SYM}_d(\mathbf{x})$ of formal degree at most $O(d)$ must have top fan-in at least $n^{\Omega(d)}$ (for a very large range of choice of d). Thus, higher formal degree indeed helps make the computation efficient.

Another recent example of the power non-homogeneous depth-3 circuits is a beautiful result of Gupta, Kamath, Kayal and Saptharishi [GKKS16, Tav15], who showed that over the field $\mathbb{C}$, there is a $\sum\prod\sum$ circuit of formal degree $\exp(O(\sqrt{n}\log n))$ and top fan-in $\exp(O(\sqrt{n}\log n))$ which computes the determinant of an $n\times n$ symbolic matrix. Prior to this work, the best $\sum\prod\sum$ circuit known for the determinant was the trivial representation of the determinant as a sum of monomials and had size $\exp(\Omega(n\log n))$. In fact, in [GKKS16, Tav15], the authors prove something stronger. They show that over the field $\mathbb{C}$, if a homogeneous n-variate polynomial of degree d can be computed by an arithmetic circuit of size s ($s\geq d$), then it can be computed by an $\sum\prod\sum$ circuit of top fan-in $\exp(O(\sqrt{d}\log s))$ and formal degree $\exp(O(\sqrt{d}\log s))$.

Thus, increasing the the formal degree of $\sum\prod\sum$ circuits helps reduce their top fan-in (and size) while preserving the expressiveness. The main motivation for this work is to explore this tradeoff between the formal degree and top fan-in better. In particular, the following question is of interest to us.

Question 1.1. *In the depth reduction results of [GKKS16], can we increase the formal degree of the resulting $\sum\prod\sum$ circuit further and obtain an $\sum\prod\sum$ circuit with a smaller top fan-in ?*

To the best of our understanding, the upper bound on the top fan-in of the $\sum\prod\sum$ circuit obtained by depth reducing a general circuit of size s computing a degree d polynomial is $s^{O(t+d/t)}$, and the formal degree is $s^{O(t)}$ [AV08, GKKS16, Tav15]. Here, $t\in[d]$. Thus, regardless of the choice of t and the resulting formal degree obtained, the top fan-in upper bound is always $s^{\Omega(\sqrt{d})}$. Before we state our results, we make a brief tour into the realm of approximative or border algebraic computation, since the notion is crucial to our results here.

1.1 Approximative or Border Complexity

We now define the *approximative or border* circuit complexity of a polynomial.

Definition 1.2. *A polynomial $P\in\mathbb{F}[\mathbf{x}]$ is said to be computable in the* border *of arithmetic circuits of size at most s iff there are polynomials $Q_1,Q_2,\ldots,Q_t\in\mathbb{F}[\mathbf{x}]$ such that the polynomial $Q\equiv P+\sum_{j=1}^{t}\varepsilon^j\cdot Q_j$ can be computed by an arithmetic circuit $C\in\mathbb{F}(\varepsilon)[\mathbf{x}]$ of size at most s, where ε is a new indeterminate.* $\diamondsuit$

An analogous definition applies when arithmetic circuits in the above definition is replaced by some other model of computation, for instance, arithmetic formula, or algebraic branching programs of a certain width. This notion of approximation is often referred to as the *algebraic* approximation, as opposed to the usual *topological* notion. For this paper, we only work with algebraic approximation upper bounds, which trivially imply upper bounds in the topological sense as well. We refer the reader to an excellent discussion on border complexity in the work of Bringmann et al. [BIZ18].

It follows from the definitions that a polynomial which can be computed by an arithmetic circuit of size s is trivially in the border of size s circuits, but in general we do not know implications in the other direction. In particular, it is potentially easier to prove border complexity upper bounds for a model, and harder to prove border complexity lower bounds. And, indeed we know some upper bounds in the border complexity framework which are either false, or not known in the exact computation framework. We discuss two such results.

Low degree factors of polynomials with small circuits. In [Bür04], Bürgisser showed that if a polynomial $P \in \mathbb{C}[\mathbf{x}]$ has circuit complexity at most s, and f is an irreducible factor of P of degree d, then f has border circuit complexity at most $\mathsf{poly}(s, d)$. In particular, this upper bound does not depend on the degree of P itself, which could be as large as 2^s! For exact computation, we only know that f can be computed by an arithmetic circuit of size $\mathsf{poly}(s, d, m)$, where m is the largest integer such that f^m divides P. Note that m can be as large as $\exp(\Omega(s))$, and hence the bound is not always polynomially bounded in s and d. Extending the results in [Bür04] continues to be a fascinating and fundamental open problem.

Width-2 algebraic branching programs. In [BIZ18], Bringmann, Ikenmeyer and Zuiddam showed that over all fields of characteristic different from 2, if a polynomial P of degree d has an arithmetic formula of size s, then P is in the border of width-2 algebraic branching programs of size at most $\mathsf{poly}(s)$. This is the approximative version of a strengthening of a classical result of Ben-Or and Cleve [BC88] who showed that if a degree d polynomial P has an arithmetic formula of size s, then P can be computed by a width-3 algebraic branching program of size $\mathsf{poly}(s)$. The result in [BIZ18] is surprising, since we know that an analog of the result of Ben-Or and Cleve is false for width-2 algebraic branching programs. In fact, Allender and Wang [AW16] showed that width-2 algebraic branching programs are not even universal, i.e. there are polynomials which they cannot compute regardless of the size.

1.2 Results

Our main result is the following theorem.

Theorem 1.3 (Approximating general polynomials)**.** *Let $P \in \mathbb{C}[\mathbf{x}]$ be any homogeneous polynomial of degree d. Then, there exists a $\sum\prod\sum$ circuit $C \in \mathbb{C}(\varepsilon)[\mathbf{x}]$ with top fan-in at most 2 and formal degree at most $\left(d \cdot 2^d \cdot \binom{n+d-1}{d-1}\right)$ such that*

$$C \equiv P + \varepsilon Q\,,$$

where $Q \in \mathbb{C}[\varepsilon, \mathbf{x}]$ and every monomial with a non-zero coefficient in Q has degree strictly greater than d.

Thus, *every* homogeneous polynomial of degree d is in the border of $\sum\prod\sum$ circuits with top fan-in at most 2. Of course, the upper bound on the formal degree is extremely high, and up to lower order terms, this is unavoidable due to counting arguments. We remark that this result is a bit surprising since it known to be false in the realm of exact computation. More formally, the following folklore lemma is well known (at least implicitly).

Lemma 1.4 (Lemma 3.2 in [CGJ+18])**.** *Any $\sum\prod\sum$ circuit computing the inner product polynomial $IP = \sum_{i=1}^{n} x_i y_i$ must have top fan-in $\Omega(n)$, regardless of the formal degree.*

Thus, the exact computation analog of Theorem 1.3 is false in a very strong sense, even for polynomials of degree 2.

Theorem 1.3 has an interesting implication with respect to proving top fan-in lower bounds for the border of $\sum\prod\sum$ circuits. It implies that any such lower bound has to rely on the formal degree of the circuit being somewhat small, else as Theorem 1.3 shows, one can compute *every* homogeneous polynomial in the border of a $\sum\prod\sum$ circuit with top fan-in two.

Our second theorem is a special case of Theorem 1.3 for sums of powers of linear forms. The upper bound on the formal degree of the approximating $\sum\prod\sum$ circuit obtained here is much better.

Theorem 1.5 (Approximating sums of powers of linear forms). *Let $P = \sum_{i=1}^{T} \ell_i^d$ be any homogeneous polynomial of degree d in $\mathbb{C}[\mathbf{x}]$, where each ℓ_i is a homogeneous linear form. Then, there exists a $\sum\prod\sum$ circuit $C \in \mathbb{C}(\varepsilon)[\mathbf{x}]$ with top fan-in at most 2 and formal degree at most $(d \cdot T)$ such that*

$$C \equiv P + \varepsilon Q\,,$$

where $Q \in \mathbb{C}[\varepsilon, \mathbf{x}]$ and every monomial with a non-zero coefficient in Q has degree strictly greater than d.

Our final result answers Question 1.1 in the affirmative in the border complexity sense. We prove the following statement.

Theorem 1.6 (Top fan-in vs formal degree for chasm at depth-3). *Let $P \in \mathbb{C}[\mathbf{x}]$ be a homogeneous polynomial of degree d which is computable by an arithmetic circuit of size $s \geq d$. Then, for every $t \in [d]$, there is a $\sum\prod\sum$ circuit $C_t \in \mathbb{C}(\varepsilon)[\mathbf{x}]$ of top fan-in at most $s^{O(t)}$ and formal degree $s^{O(d/t)}$ such that*

$$C_t \equiv P + \varepsilon Q\,,$$

where $Q \in \mathbb{C}[\varepsilon, \mathbf{x}]$.

As remarked earlier, this tradeoff is in contrast to the original result of Gupta, Kamath, Kayal and Saptharishi [GKKS16] where the top fan-in is always at least $s^{\Omega(\sqrt{d})}$ regardless of the formal degree of the circuit. This appears to be a consequence of the fact that the polynomial $(\sum_{i=1}^{n} x_i)^d$ cannot be expressed as a sum of product of univariates where the number of summands depends only on d.

In the rest of this note, we include the proofs of the above theorems. All our proofs are based on very simple and elementary ideas building on top of known results in this area, most notably those in [AV08, Tav15, Fis94]. However, the theorem statements (especially Theorem 1.3) seem to be interesting (and surprising!).

2 Preliminaries

- Unless otherwise stated, we work over the field $\mathbb{C}$ of complex numbers.
- n is the number of variables and d is the degree.
- We use boldface letters like $\mathbf{x}$ to denote the set of variables $\{x_1, x_2, \ldots, x_n\}$.
- For a natural number $m > 0$, $[m]$ denotes the set $\{1, \ldots, m\}$.
- For a scalar $\alpha \in \mathbb{F}$, and a polynomial $P \in \mathbb{F}[\mathbf{x}]$, $P(\alpha \cdot \mathbf{x}) = P(\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_n)$.
- The sparsity of a polynomial is the number of monomials with non-zero coefficients in the polynomial.
- For an indeterminate ε and a field $\mathbb{F}$, $\mathbb{F}(\varepsilon)$ is the field of rational functions in ε over $\mathbb{F}$.

2.1 Fischer's Identity

Our main tool is the following classical lemma of Fischer [Fis94](see also [Shp02]).

Lemma 2.1 (Fischer [Fis94], Shpilka [Shp02]). *Let $Q \in \mathbb{C}[\mathbf{x}]$ be a monomial of degree d. Then, there exist homogeneous linear forms $\ell_1, \ell_2, \ldots, \ell_{2^d} \in \mathbb{C}[\mathbf{x}]$ such that*

$$Q = \sum_{i=1}^{2^d} \ell_i^d .$$

The lemma immediately extends to general homogeneous polynomials in the following sense.

Corollary 2.2. *Let $P \in \mathbb{C}[\mathbf{x}]$ be any homogeneous polynomial of degree d. Then, there exist linear forms $\ell_1, \ell_2, \ldots, \ell_m \in \mathbb{C}[\mathbf{x}]$ for $m \leq \left(2^d \cdot \text{Sparsity}(P)\right)$, such that*

$$P = \sum_{i=1}^{m} \ell_i^d .$$

3 Proofs of main theorems

3.1 Approximating general homogeneous polynomials.

Proof of Theorem 1.3. Let $P \in \mathbb{C}[\mathbf{x}]$ be any homogeneous degree d polynomial. Then, from Corollary 2.2, we know that for $m \leq \left(2^d \cdot \text{Sparsity}(P)\right)$ there are homogeneous linear forms $\ell_1, \ell_2, \ldots, \ell_m$ such that

$$P = \sum_{i=1}^{m} \ell_i^d .$$

If the polynomial A is defined as $A = \prod_{i=1}^{m}(1 + \ell_i^d)$, then A is of the following form:

$$A = 1 + (\sum_{i=1}^{m} \ell_i^d) + B ,$$

where every monomial in B has degree at least $2d$. Moreover, the second term $\sum_{i=1}^{m} \ell_i^d$ is just equal to P. Thus, we have,

$$A = 1 + P + B .$$

Moreover, since the field $\mathbb{C}$ is algebraically closed, the polynomial $(1 + \ell_i^d)$ factors into a product of affine forms. Thus, the polynomial A can be written as a product of linear forms, $(1 + \ell_i^d) = \prod_{j=1}^{d}(\alpha_{i,j} + \ell_i)$ for appropriate complex numbers $\{\alpha_{i,j} : i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, d\}\}$. Therefore, $1 + P + B$ is a product of at most md affine forms.

Now, replacing every x_i by εx_i, we get

$$1 + \varepsilon^d P + \varepsilon^{2d} R = \prod_{i=1}^{m} \prod_{j=1}^{d} (\alpha_{i,j} + \varepsilon \ell_i) .$$

After rearranging and rescaling, we get

$$P + \varepsilon^d R = -\varepsilon^{-d} + \varepsilon^{-d} \cdot \prod_{i=1}^{m} \prod_{j=1}^{d} (\alpha_{i,j} + \varepsilon \ell_i) .$$

The right hand side is clearly a sum of two product of linear forms over the field $\mathbb{C}(\varepsilon)$. This completes the proof of the theorem. □

Approximating sums of powers of linear forms.

Proof of Theorem 1.5. The proof of Theorem 1.5 is essentially the same as the proof of Theorem 1.3, except that instead of using the bound of $\left(2^d \text{Sparsity}(P)\right)$ on the number of summands in a sum of power of linear forms representation of the polynomial, we use the bound of T given in the hypothesis of this theorem. Thus the final upper bound on the formal degree that we get here is $d \cdot T$.

□

3.2 Approximate reduction to depth-3.

Proof of Theorem 1.6. For this proof, we will follow the outline in [GKKS16], but the argument is considerably simpler. Let $t \in [d]$ be a parameter. Let P be a homogenous polynomial of degree d computable by a size s arithmetic circuit. We first transform this homogeneous circuit of unbounded depth to a homogeneous $\sum^{[T]} \prod^{[O(t)]} \sum^{[M]} \prod^{[d/t]}$ circuit using the standard depth reduction to depth-4 [AV08, Tav15]. Here the superscripts in square brackets indicate an upper bound on the fan-in of gates in the corresponding layer of the depth-4 circuit. It follows from [AV08, Tav15] that $T \leq s^{O(t)}$ and $M \leq s^{O(d/t)}$.

Now, we apply Theorem 1.3 to every $\sum^{[M]} \prod^{[d/t]}$ sub-circuit in this homogeneous depth-4 circuit. Each of these homogeneous polynomials is a sum of at most $M \leq s^{O(d/t)}$ monomials of degree at most d/t. We replace each of these polynomials by a depth-3 circuit of top fan-in 2 and formal degree $d/t \cdot 2^{d/t} \cdot s^{O(d/t)}$, given by Theorem 1.3, which approximates the polynomial (in the border complexity sense). Thus, we end up with a $\sum^{[T]} \prod^{[O(t)]} \sum^{[2]} \prod^{[(d/t)\cdot 2^{(d/t)} \cdot s^{O(d/t)}]} \sum^{[n+1]}$ circuit which approximates P. Now, converting each $\prod^{[O(t)]} \sum^{[2]}$ sub-circuit into a $\sum^{[2^{O(t)}]} \prod^{[O(t)]}$ circuit by distributivity, we get a $\sum^{[s^{O(t)} \cdot 2^{O(t)}]} \prod^{[O(t)\cdot(d/t)\cdot 2^{(d/t)} \cdot s^{O(d/t)}]} \sum^{[n+1]}$ circuit which approximates P. Thus we get a depth-3 circuit with top fan-in at most $s^{O(t)}$ and the formal degree is at most $s^{O(d/t)}$ which approximates P in the border complexity sense.

□

3.3 Partial interpolation

A standard and widely used trick in algebraic complexity is the use of univariate polynomial interpolation to extract homogeneous components of a polynomial which can be computed by a small arithmetic circuit. In this section, we discuss an analog of this in the border complexity setting, which is slightly more efficient than the standard interpolation arguments for an appropriate range of parameters. In an earlier version of this paper, this lemma was used to give a weaker version of Theorem 1.3, where the top fan upper bound was $d + 1$ for a degree d polynomial as opposed to the current bound of 2 which is independent of the degree of the polynomial. Nevertheless, we think that this lemma might be of independent interest.

Lemma 3.1 (Approximating low degree homogeneous components)**.** *Let $\mathbb{F}$ be any field of size at least $d+1$ and let $P(\mathbf{x}) = \sum_{i=0}^{d} P_i(\mathbf{x})$ be a polynomial of degree d in $\mathbb{F}[\mathbf{x}]$ where for each $i \in \{0, 1, \ldots, d\}$, $P_i(\mathbf{x})$ is the homogeneous component of P of degree equal to i. Then, for every $i \in \{0, 1, \ldots, d\}$ and for every choice of $i+1$ distinct elements $\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,i}$ in $\mathbb{F}$, there exist $\beta_{i,0}, \beta_{i,1}, \ldots, \beta_{i,i}$ in $\mathbb{F}$ such that*

$$\sum_{j=0}^{i} \beta_{i,j} \cdot P(\alpha_{i,j} \cdot \mathbf{x}) = P_i(\mathbf{x}) + R\,,$$

where the degree of every monomial with a non-zero coefficient in R is at least $i+1$.

Proof. Let y be a new formal variable, and let $Q(y) \in (\mathbb{F}[\mathbf{x}])[y]$ be the defined as

$$Q(y) = P(yx_1, yx_2, \ldots, yx_n)\,.$$

Clearly, $Q(y) = \sum_{j=0}^{d} y^j P_j(\mathbf{x})$. For the rest of the proof, we fix an arbitrary $i \in \{0, 1, \ldots, d\}$. Let $\alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,i}$ be any $i+1$ distinct elements of $\mathbb{F}$. Then, for every $k \in \{0, 1, \ldots, j\}$, we have

$$Q(\alpha_{i,k}) = \sum_{j=0}^{d} \alpha_{i,k}^{j} P_j(\mathbf{x})\,.$$

For $j \in \{0, 1, \ldots, i\}$, let $\gamma_j = (\alpha_{i,j}^0, \alpha_{i,j}^1, \alpha_{i,j}^2, \ldots, \alpha_{i,j}^i)$. From the choice of $\alpha_{i,j}$, we know that for any $j \neq j'$, $\alpha_{i,j} \neq \alpha_{i,j'}$. Thus, it follows that $\gamma_0, \gamma_1, \ldots, \gamma_i$ are linearly independent, and hence there exist scalars $\beta_{i,0}, \beta_{i,1}, \ldots, \beta_{i,i}$ in $\mathbb{F}$ such that

$$\sum_{j=0}^{i} \beta_{i,j}\gamma_j = (0, 0, \ldots, 1)\,. \tag{3.2}$$

Therefore,

$$\begin{aligned}
\sum_{j=0}^{i} \beta_{i,j} Q(\alpha_{i,j}) &= \sum_{j=0}^{i} \beta_{i,j} \left(\sum_{j'=0}^{d} \alpha_{i,j}^{j'} P_{j'}(\mathbf{x}) \right) \\
&= \sum_{j'=0}^{d} \left(\sum_{j=0}^{i} \beta_{i,j} \alpha_{i,j}^{j'} \right) P_{j'}(\mathbf{x})
\end{aligned}$$

From Equation 3.2, we know that for every $j' < i$,

$$\sum_{j=0}^{i} \beta_{i,j} \alpha_{i,j}^{j'} = 0\,,$$

and,

$$\sum_{j=0}^{i} \beta_{i,j} \alpha_{i,j}^{i} = 1\,.$$

Thus,

$$\sum_{j=0}^{i} \beta_{i,j} Q(\alpha_{i,j}) = P_i(\mathbf{x}) + (\text{monomials of degree} > i)\,.$$

□

4 Open problems

We end with some open problems.

- Perhaps the most natural question is to understand if the *exact computation* versions of Theorem 1.3, Theorem 1.5 or Theorem 1.6 are true with a reasonable blow up in the parameters. For instance, can every homogeneous polynomial of degree d be computed by a $\sum\prod\sum$ circuit with top fan-in $\mathsf{poly}(n)$, if arbitrarily large formal degree is allowed ? What about all polynomials in VP ?

- Another question is to understand if there are natural classes of polynomials for which the formal degree upper bound in Theorem 1.3 can be reduced to some more reasonable (and possibly useful) upper bound, while keeping the top fan-in small. For instance, can every homogeneous degree d polynomial with a formula of size s be approximated by a $\sum\prod\sum$ circuit of top fan-in $\mathsf{poly}(d)$ and formal degree $\mathsf{poly}(s, d)$?

- And finally, Theorem 1.3 seems to be a very general structural statement for low degree polynomials. Does this have other applications ?

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