Stopping by Matrix Rigidity on a snowy day

Introduction to Matrix Rigidity - I

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Mumbai, INDIA

Workshop on Matrix Rigidity
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- Rigidity intertwines *combinatorial* & *algebraic* property.
- Rigidity has connections to communication complexity, data structure lower bounds and coding theory.

Interpreting Matrix Rigidity

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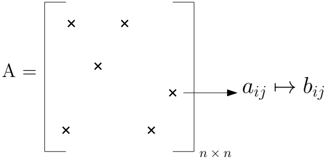

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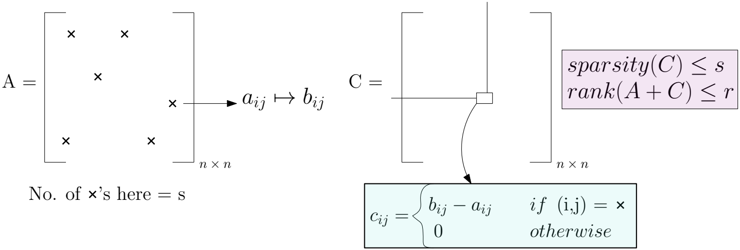

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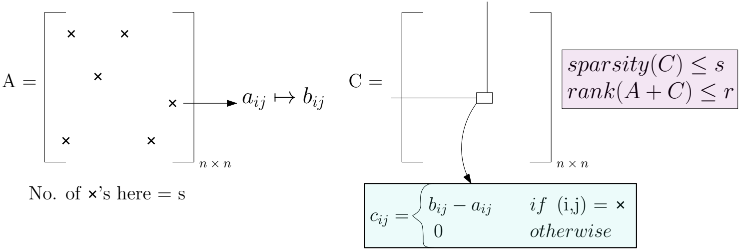

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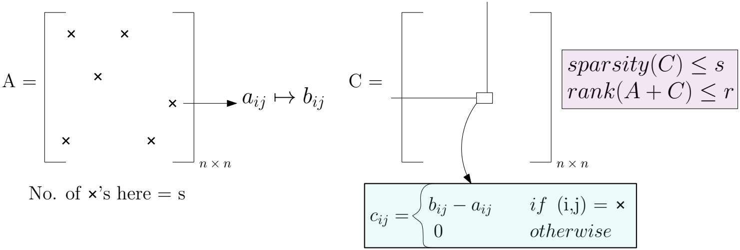

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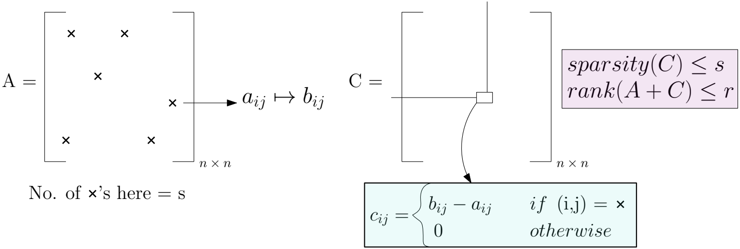

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Rigidity of a matrix A for rank r

$$R_A^{\mathbb{F}}(r) = \min_C \{\text{sparsity(C)} \mid C \in \mathbb{F}^{n\times n},\ \text{rank}_{\mathbb{F}}(A + C) \leq r\}.$$

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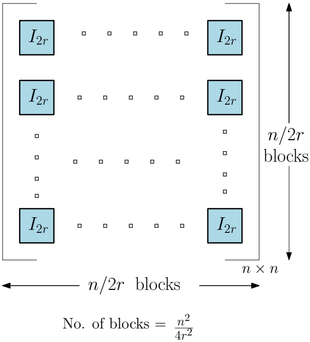
$$\text{rank}(I_n + C) \geq \text{rank}(I_n) - \text{rank}(C) \geq n - (n - r) > r (\Leftarrow\!\!\Rightarrow)$$

Toy Example II: Building over Identity matrices

Theorem (Midrijānis (2005))

For any n divisible by $2r$, $R^{\mathbb{F}}_{M_n}(r) = \frac{n^2}{4r}$.



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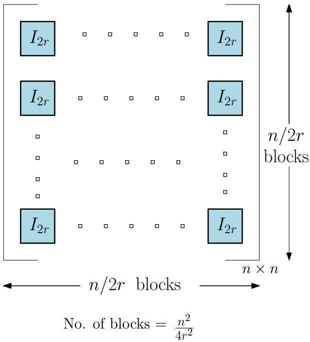

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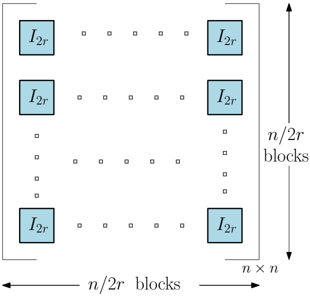
- Clearly, by changing r entries in each block *consistently* $\text{rank}(M_n) \leq r$. Thus, $R_{M_n}(r) \leq \frac{n^2}{4r}$.
- Suppose, $rank(M_n)$ can be reduced to r by changing fewer than $\frac{n^2}{4r}$ entries. Then, $\exists\ I_{2r}$ block whose rank can be reduced to r by changing fewer than r entries. $(\Leftarrow\Rightarrow)$



No. of blocks $= \frac{n^2}{4r^2}$

Upper Bounds on Matrix Rigidity

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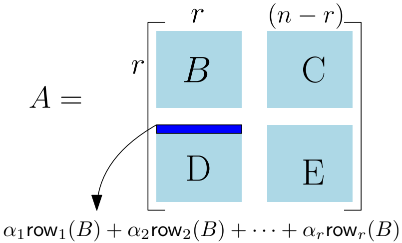

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- Edit every row of E by corresponding linear combination of the r rows of C.

$$A = \begin{array}{cc} & \begin{array}{cc} r & (n-r) \end{array} \\ r & \begin{bmatrix} B & C \\ D & E \end{bmatrix} \end{array}$$

$$\alpha_1 \mathrm{row}_1(C) + \alpha_2 \mathrm{row}_2(C) + \cdots + \alpha_r \mathrm{row}_r(C)$$

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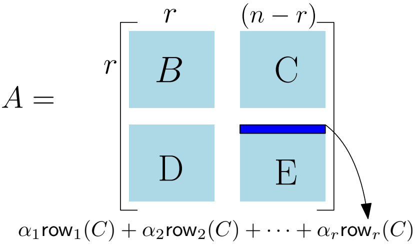

Now, every row of A is a linear combination of the first r rows.
By changing $(n-r)^2$ entries in E, $\text{rank}(A)$ is reduced to r.
Thus, $R_A^{\mathbb{F}}(r) \leq (n-r)^2$.

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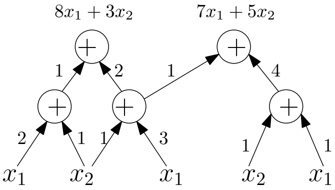

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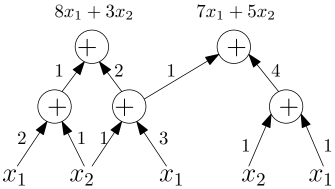

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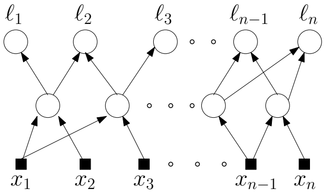

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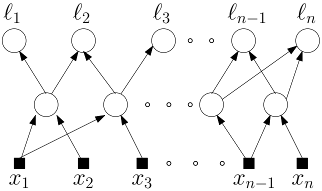

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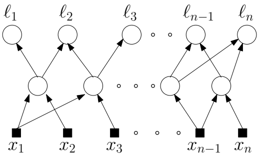

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> - size($\mathcal{C}$): # of edges
> - depth($\mathcal{C}$): length of longest path from i/p to o/p.

- Any linear transformation $\mathbb{F}^n \to \mathbb{F}^n$ can be computed by a linear circuit of size $O(n^2)$ and depth $O(\log n)$.

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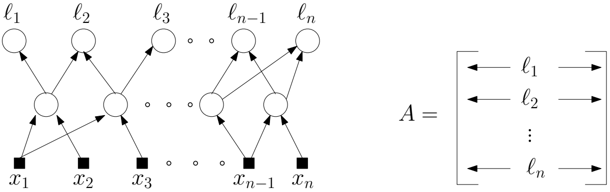

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- Best known size lower bound: $3n - o(n)$ (Chashkin 1994).

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- *Rigid* matrices cannot be computed by linear circuits having *small* depth as well as *small* size.

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- Repeating the *edge removal process* ϵ times, length of every path at most $d/2^{\epsilon}$ and no. of edges removed is $\frac{s\epsilon}{\log d}$.

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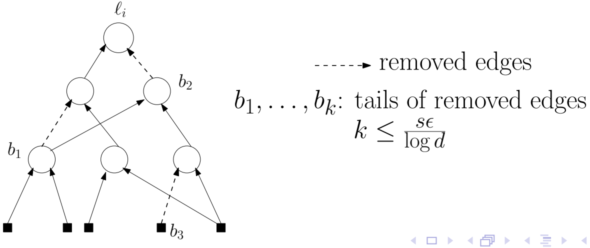
Let G be a directed acyclic graph with s edges and every path having length at most d. Then, by removing at most $s/\log d$ edges every path in the resulting graph has length at most $d/2$.

- Repeating the *edge removal process* ϵ times, length of every path at most $d/2^{\epsilon}$ and no. of edges removed is $\frac{s\epsilon}{\log d}$.



Proof(contd.)

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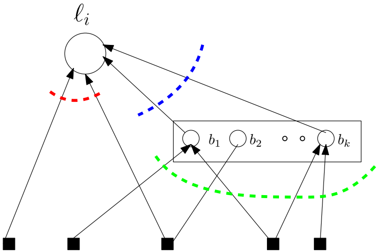

$$\ell_i = \Sigma_{j=1}^{k} \alpha_{ij} b_j + c_i$$

$$\alpha_{ij} \in F \;\; b_j \in F^n$$

$$c_i \in F^n, \; 2^{d/2^\epsilon}\text{-sparse}$$

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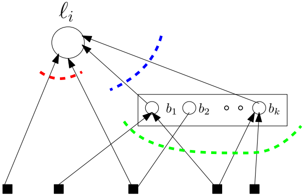

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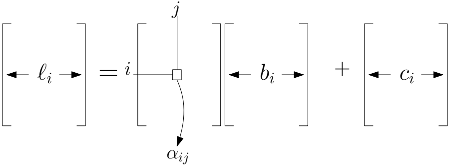

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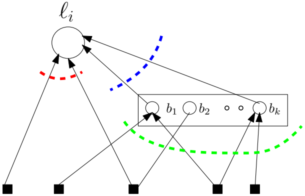

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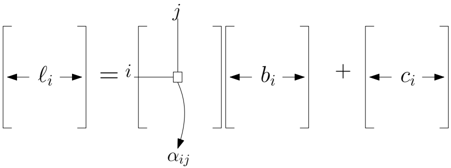

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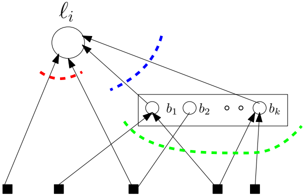

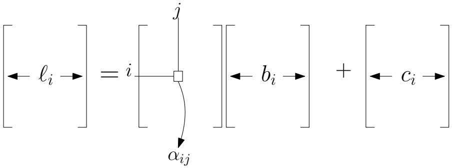

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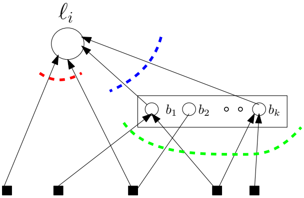

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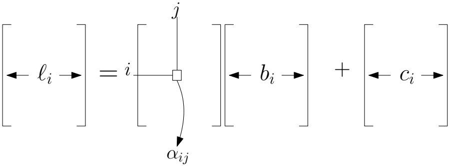

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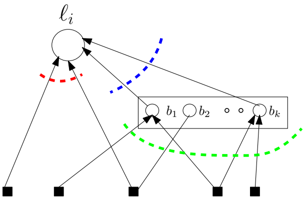

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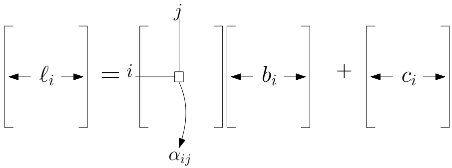

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- If $A \in \mathbb{F}^{n \times n}$ is computed by a linear circuit of size $n \log\log n$ and depth $\log n$ then $R_A(\epsilon n) \leq n^{1+\delta}$.

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- For any $A \in \mathbb{F}^{n \times n}$ if $R_A(\epsilon n) > n^{1+\delta}$ for some $\epsilon, \delta > 0$ then any linear circuit of depth $O(\log n)$ computing A must have size $\Omega(n \log \log n)$.

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This Workshop

Recent Progress towards answering Valiant's Question (and beyond).

Existence of Rigid Matrices

Theorem (Valiant(1977))

Let $\mathbb{F}_q$ *be a finite field. For any* $0 \leq r \leq n - \Omega(\sqrt{n})$ *there is a matrix* $M \in \mathbb{F}_q^{n \times n}$ *such that* $R_M^{\mathbb{F}_q}(r) = \Omega((n-r)^2 / \log n)$.

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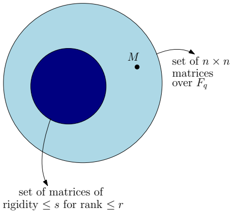

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- When $r < n - c_1\sqrt{n}$ and $s < c_2(n-r)^2/\log n$ almost all matrices have rigidity $(n-r)^2$.

Super-exponential time construction of Rigid Matrices

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- For every $n \times n$ matrices A with entries in $\mathbb{F}_q$, test if there exists any s-sparse matrix C such that $\mathrm{rank}_{\mathbb{F}_q}(A + C) \leq r$.
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Theorem (Valiant(1977))

Let $\mathbb{F}$ be an infinite field. For any $0 \leq r \leq n$ there is a matrix $M \in \mathbb{F}^{n \times n}$ such that $R_M^{\mathbb{F}}(r) = (n - r)^2$.

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Theorem (Shokrollahi, Spielman, Stemann(1997))

Let $\mathbb{F}$ be a field with at least $2n$ distinct elements and M_n be $n \times n$ Cauchy matrix. Then, $R_{M_n}^{\mathbb{F}}(r) = \Omega(\frac{n^2}{r} \log \frac{n}{r})$ for $\log n \leq r \leq n/2$.

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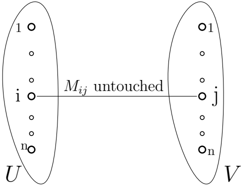
- G has at least $n^2 - o(\frac{n^2}{r}\log\frac{n}{r})$ edges.



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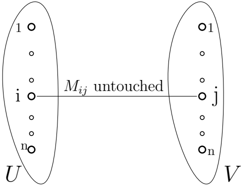

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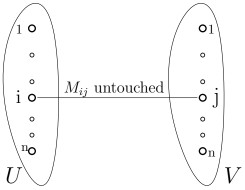

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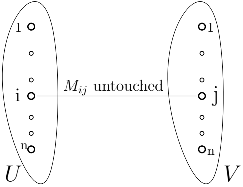

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- If fewer than $\frac{n^2}{4(r+1)}\log\frac{n}{r}$ entries in M are changed an $(r+1) \times (r+1)$ submatrix of M_n remains untouched.

Matrices with Algebraically Independent Entries

- $a_1, \ldots, a_n \in \mathbb{R}$ are *algebraically independent* over $\mathbb{Q}$ if there is no polynomial $P \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $P(a_1, \ldots, a_n) = 0$.

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Theorem

Let $A \in \mathbb{R}^{n \times n}$ with n^2 algebraically independent elements over $\mathbb{Q}$ as its entries. Then, for any $r \leq n$, $R_A^{\mathbb{R}}(r) = (n-r)^2$.

Proof. Upper bound via Valiant's theorem.

Matrices with Algebraically Independent Entries

Lower Bound: Suppose not, $R_A^{\mathbb{R}}(r) < (n-r)^2$. Then $A = S + L$ such that S has sparsity $s < (n-r)^2$ and L has rank $\leq r$.

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- Can we reduce the amount of algebraic independence among the entries while maintaining rigidity?

Non-explicit Rigid Matrices

Theorem (Lokam(2000, 2006))

- *Let* $x_1, \ldots, x_n \in \mathbb{C}$ *be algebraically independent over* $\mathbb{Q}$ *and* $V = (x_i^j)_{1 \le i,j \le n}$ *be Vandermonde matrix in* $\mathbb{C}^{n \times n}$*. Then,* $R_V^{\mathbb{C}}(r) = \Omega(n^2)$ *for* $r \le O(\sqrt{n})$*.*

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- Proof via *algebraic dimension* argument(Shoup, Smolensky).

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[Goldreich, Tal 2013] Rigidity of Random Toeplitz matrix

For every $r \in [\sqrt{n}, n/32]$, $\mathcal{R}_T^{\mathbb{F}_2}(r) = \Omega\left(\frac{n^3}{r^2 \log n}\right)$ with probability $1 - o(1)$ where $T \in \mathbb{F}_2^{n \times n}$ is a random Toeplitz/Hankel matrix.

Toeplitz $T = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_{-1} & a_0 & a_1 \\ a_{-2} & a_{-1} & a_0 \end{bmatrix}$ and Hankel $H = \begin{bmatrix} a_{-2} & a_{-1} & a_0 \\ a_{-1} & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix}$

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Explicit construction in E^{NP}

- Run over all $n \times n$ Hankel/Toeplitz matrices with $\{0, 1\}$ entries.
- For each such matrix test if $\mathcal{R}_T^{\mathbb{F}_2}(r) = \Omega\left(\frac{n^3}{r^2 \log n}\right)$.

Designing $\mathrm{TEST}_{s,r}(H)$

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(1) If H is not rigid then reject H.
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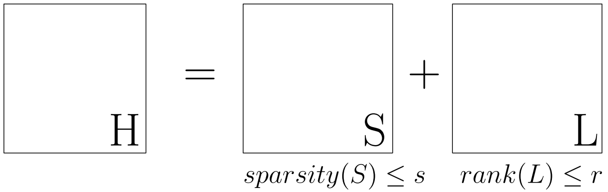

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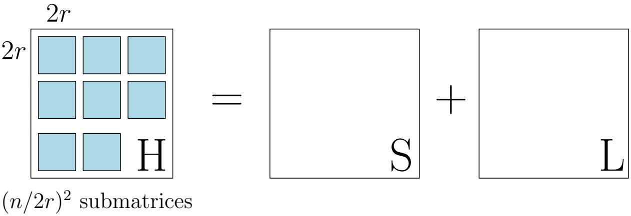

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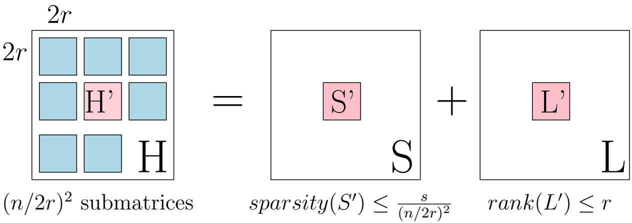

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For every such submatrix H' of H
  For every $\frac{s}{(n/2r)^2}$-sparse matrix S' in $\mathbb{F}_2^{2r \times 2r}$
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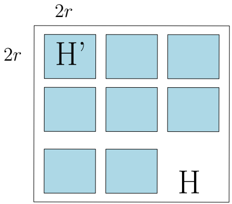
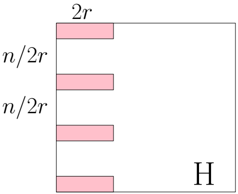


$(n/2r)^2$ submatrices



Explicit Rigid matrices beyond exponential time

- (Folklore) Sub-exponential time construction of $M \in \mathbb{F}_2^{n \times n}$ with $R_M^{\mathbb{F}_2}(n^{1/2-\epsilon}) \geq \Omega(n^2/\log n)$.
- (Alman, Chen '20) $M \in \mathbb{F}_2^{n \times n}$ in P^{NP} such that there exists a $\delta > 0$ with $R_M(2^{(\log n)^{1/4-\epsilon}}) \geq \delta n^2$ for all $\epsilon > 0$.
- (Bhangale, Harsha, Paradise, Tal '20) $M \in \mathbb{F}_2^{n \times n}$ in P^{NP} such that there exists a $\delta > 0$ with $R_M(2^{\log n/\Omega(\log\log n)}) \geq \delta n^2$.

- Works for any finite field for large n.
- Proof via linear circuit lower bounds & PCPs.

The Road Thus Taken

1977	$R_A(\epsilon n) = n^{1+\delta}$	GOAL
1977	$\exists M\ R_A(r) = (n-r)^2/\log n$	Existence
1997	A in P, $R_A(r) = \Omega\left(\frac{n^2}{r}\log\frac{n}{r}\right)$	Untouched Minor argument
2000,2006	Non-explicit $R_A(r) = \Omega(n^2)$	Shoup-Smolensky Dimension
2013	A in E^{NP}, $R_A(r) = \Omega\left(\frac{n^3}{r^2 \log n}\right)$	Rigidity of Random Toeplitz matrices
2020	A in $2^{o(n)}$-time, $R_A(n^{0.5-\epsilon}) = \Omega\left(\frac{n^2}{\log n}\right)$	Talk by Ben Lee Volk
2020	A in P^{NP}, $R_A\left(2^{\log n/\Omega(\log\log n)}\right) \geq \delta n^2$	Talk by Amey Bhangale

Thank You! Questions?